Math 255A' Lecture 2 Notes

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1 More Hilbert Space Review

1.1 Linear functionals

Let H be a Hilbert space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We want to consider linear functionals $L : H \to \mathbb{F}$.

Proposition 1.1. *Let $L : H \to \mathbb{F}$ be linear. The following are equivalent.*

1. *L is continuous.*
2. *L is continuous at 0.*
3. *L is continuous at one point.*
4. *L is bounded ($\exists c < \infty$ such that $|L(h)| \le c\|h\|$ for all $h \in H$).*

Definition 1.1. For a bounded linear functional L its **norm** is

$$\begin{aligned} \|L\| &= \inf\{c > 0 : |L(h)| \le c\|h\|\} \\ &= \sup\left\{\frac{|L(h)|}{\|h\|} : h \in H \setminus \{0\}\right\} \\ &= \sup\{|L(h)| : \|h\| = 1\}. \end{aligned}$$

Theorem 1.1 (Riesz representation)**.** *If $L : H \to \mathbb{F}$ is a bounded linear functional, then there is a unique $h_0 \in H$ such that $L(h) = \langle h, h_0 \rangle$ for all $h \in H$. Moreover, $\|L\| = \|h_0\|$.*

Corollary 1.1. *If $L : L^2_{\mathbb{R}}(\mu) \to \mathbb{R}$ is a bounded linear functional, then there exists a unique $h_0 \in L^2_{\mathbb{R}}(\mu)$ such that $L(h) = \int h\overline{h}_0 \, d\mu$ for all $h \in L^2_{\mathbb{R}}(\mu)$.*

1.2 Orthonormal sets and bases

Definition 1.2. A subset $\mathcal{E} \subseteq H$ is **orthonormal** if $\langle e, e' \rangle = \delta_{e,e'}$ for all $e, e' \in \mathcal{E}$. $\mathcal{E}$ is a **basis** if it is maximal under inclusion.

Proposition 1.2. *Any orthonormal set is contained in a basis.*

The proof uses Zorn's lemma.[1]

Example 1.1. In $L^2_{\mathbb{C}}([0, 2\pi])$, let $e_n(t) = \frac{1/\sqrt{2\pi}}{e}^{int}$. The set $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal set (it is actually a basis, too).

Example 1.2. In $\mathbb{F}^n$, let e_k be the vector with all 0s except a 1 in the k-th coordinate. Then $\{e_1, \ldots, e_n\}$ is an orthonormal basis.

Example 1.3. In $\ell^2 = \{(x_i)_{i=1}^{\infty} : \sum_i |x_i|^2 < \infty\}$, let e_n be the vector with all 0s except a 1 in the n-th coordinate. Then $\{e_n : n \in \mathbb{N}^+\}$ is an orthonormal basis.

Theorem 1.2 (Gram-Schmidt procedure)**.** *If $(h_n)_{n \geq 1}$ is linearly independent, then there is an orthonormal sequence $(e_n)_{n \geq 1}$ such that for all $N \in \mathbb{N}$, we have s*$\operatorname{span}\{h_1, \ldots, h_N\} = \operatorname{span}\{e_1, \ldots, e_N\}$.

Proposition 1.3. *Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in H, and let their span be $M = \operatorname{span}\{e_1, \ldots, e_n\}$. Then $P_M h = \sum_{i=1}^{n} \langle h, e_i \rangle e_i$.*

Proof. Recall that $P_M h$ is the unique vector in M such that $h - P_M h \perp M$. Check this property. $\square$

Theorem 1.3 (Bessel's inequality)**.** *If $(e_n)_{n \geq 1}$ is an orhonormal sequence in H and $h \in H$, then $\sum_{i \geq 1} | \langle h, e_n \rangle |^2 \leq \|h\|^2$.*

Proof. Fix $n \in \mathbb{N}$. Then consider $\langle h, e_1 \rangle, \cdots, \langle h, e_n \rangle e_n, h - P_n h$. The Pythagorean identity gives $\sum_{i=1}^{n} | \langle h, e_i \rangle |^2 + \|h - P_n h\|^2 = \|h\|^2$. Removing the term $\|h - P_n h\|^2$ gives the inequality for n. $\square$

Corollary 1.2. *If $\mathcal{E}$ is an orthonormal set in H and $h \in H$, then $\mathcal{E}_0 = \{e \in \mathcal{E} : \langle h, e \rangle \neq 0\}$ is countable.*

Proof. We have $\mathcal{E}_0 = \bigcup_{n \geq 1} \mathcal{E}_n$, where $\mathcal{E}_n = \{e \in \mathcal{E} : | \langle h, e \rangle | \geq 1/n\}$. So Bessel's inequality implies $|\mathcal{E}_n| \leq n^2 \|h\|^2$. In particular, each $\mathcal{E}_n$ is finite. $\square$

Corollary 1.3. *If $\mathcal{E}$ is orthonormal in H and $h \in H$, then*

$$\sum_{e \in \mathcal{E}} | \langle h, e \rangle |^2 \leq \|h\|^2.$$

[1] You cannot do this without waving the magic set theory wand.

Remark 1.1. By the sum over all $e \in \mathcal{E}$, we mean that it is a countable sum, since all but countably many terms in the sum are 0 for each $h \in H$.

What if we want to talk about uncountable sums in general?

Definition 1.3. Let $(h_i)_{i\in I}$ be an indexed family in H. Then

$$\sum_{i\in I} h_i = k$$

means that for every $\varepsilon > 0$, there is a finite $F \subseteq I$ such that whenever $F \subseteq F \subseteq I$ and $|G| < \infty$, we have $\|k - \sum_{i\in G} h_i\| < \varepsilon$.[2]

Lemma 1.1. *If $\mathcal{E}$ is an orthonormal set in H, $M = \overline{\operatorname{span}}\,\mathcal{E}$, and $\mathbb{P} = P_M$, then*

$$Ph = \sum_{e\in\mathcal{E}} \langle h, e\rangle\, e.$$

Theorem 1.4. *Let $\mathcal{E}$ be an orthonormal set in M. The following are equivalent:*

1. *$\mathcal{E}$ is a basis*
2. *If $h \perp \mathcal{E}$, then $h = 0$.*
3. *$\overline{\operatorname{span}}\,\mathcal{E} = H$*
4. *For all $h \in H$, $h = \sum_e \langle h, e\rangle\, e$.*
5. *For all $g, h \in H$, $\langle g, h\rangle = \sum_e \langle g, e\rangle \langle e, h\rangle$.*
6. *For all $h \in H$, $\|h\|^2 \sum_e |\langle h, e\rangle|^2$.*

Corollary 1.4. *Any two bases of H have the same cardinality.*

Definition 1.4. The **dimension** $\dim H$ is the cardinality of a basis of H.

Proposition 1.4. *An infinite-dimensional Hilbert space is separable if and only if its dimension is* $\dim H = \aleph_0$.

1.3 Isomorphisms and isometries

Definition 1.5. An **isomorphism** $A : H \to K$ is a surjective linear operator such that

1. $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in H$
2. A is surjective.

[2] This can be rephrased in terms of nets. Let's not do that.

If A only satisfies 1, it is called an **isometry**.

Example 1.4. $A : \ell^2 \to \ell^2$ sending $A(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ is an isometry but not an isomorphism.

Proposition 1.5. *A is an isometry if and only if $\|Ax\|_K = \|x\|_H$ or all $x \in H$*

Proof. ($\implies$) follows from the definition. To get ($\impliedby$), use the Polar identity. ☐

Theorem 1.5. $\dim H = \dim K$ *if and only if H is isomorphic to K.*

Proof. ($\implies$) Let $\mathcal{E}$ be a basis for H. Then define $A : H \to \ell^2(\mathcal{E})$ as $h \mapsto (\langle h, e\rangle)_{e\in\mathcal{E}}$. We get

$$\begin{tikzcd} H \arrow[d, leftrightarrow] & K \arrow[d, leftrightarrow] \\ \ell^2(\mathcal{E}) \arrow[r, leftrightarrow] & \ell^2(\mathcal{F}) \end{tikzcd}$$

☐

Corollary 1.5. *An infinite-dimensional Hilbert space is separable if and only if it is isomorphic to $\ell^2(\mathbb{N})$.*

Example 1.5. The Fourier transform is an isomorphism $L^2_{\mathbb{C}}[0, 2\pi) \to \ell^2_{\mathbb{C}}(\mathbb{Z})$ sending $f \mapsto \int_0^{2\pi} fe_n\, dt$.

1.4 Direct sums

Definition 1.6. Let H, K be inner product spaces. The **direct sum** $H \times K$ is an innerproduct space with coordinatewise addition and the inner product $\langle h \oplus k, h' \oplus k'\rangle = \langle h, h'\rangle + \langle k, k'\rangle$. For an arbitrary family $(H_i)_{i\in I}$, we define

$$\bigoplus_{i\in I} H_i = \left\{ (h_i)_{i\in I} \in \prod_{i\in I} h_i : \sum_{i\in I} \|h_i\|^2 < \infty \right\}, \qquad \langle (h_i)_i, (k_i)_i\rangle = \sum_i \langle h_i, k_i\rangle .$$